HOSSAM GHANEM

(41) 5.3 THE FUNDAMENTAL THEOREM OF CALCULUS (A)

Example 1	Find $\int_{1}^{3} (3x^2 - 2)\ dx$

Solution

$$I = \int_{1}^{3} (3x^2 - 2)\ dx = \Big[x^3 - 2x\Big]_{1}^{3} = 27 - 6 - (1 - 2) = 21 - (-1) = 21 + 1 = 22$$

Example 2	Find $\int_{-2}^{1} (2x + 3)\ dx$

Solution

$$I = \int_{-2}^{1} (2x + 3)\ dx = \Big[x^2 + 3x\Big]_{-2}^{1} = 1 + 3 - (4 - 6) = 4 - (-2) = 4 + 2 = 6$$

Example 3 35 August 15, 2009	Find $\int_{0}^{3} \frac{x}{\sqrt{16 + x^2}}\ dx$

Solution

$$I = \int_{0}^{3} \frac{x}{\sqrt{16 + x^2}}\ dx = \int_{0}^{3} (16 + x^2)^{\frac{-1}{2}} \cdot x\ dx$$

$t = 16 + x^2 \qquad dt = 2x\,dx \qquad \frac{1}{2}dt = x\,dx$

$x = 0 \quad \rightarrow \quad t = 16$

$x = 3 \quad \rightarrow \quad t = 16 + 9 = 25$

$$I = \frac{1}{2}\int_{16}^{25} t^{\frac{-1}{2}}\ dt = \frac{1}{2}\Big[2t^{\frac{1}{2}}\Big]_{16}^{25} = \frac{1}{2} \cdot 2\Big[\sqrt{t}\Big]_{16}^{25} = \sqrt{25} - \sqrt{16} = 5 - 4 = 1$$

Example 4 10 June 6, 1994	Evaluate the following integral $\int_1^2 x\sqrt{1+x^2}\,dx$

Solution

$$I = \int_1^2 x\sqrt{1+x^2}\,dx = \int_1^2 (1+x^2)^{\frac{1}{2}} \cdot x\,dx$$

$t = 1 + x^2 \qquad dt = 2x\ dx \qquad \frac{1}{2}dt = x\ dx$

$x = 1 \quad \rightarrow \quad t = 2$

$x = 2 \quad \rightarrow \quad t = 5$

$$I = \frac{1}{2}\int_2^5 t^{\frac{1}{2}}\,dt = \frac{1}{2}\left[\frac{2}{3}t^{\frac{3}{2}}\right]_2^5$$

Example 5 9 January 1994	Evaluate the following integral $\int_0^1 \frac{x^2}{(7x^3+1)^{\frac{1}{3}}}dx$

Solution

$$I = \int_0^1 \frac{x^2}{(7x^3+1)^{\frac{1}{3}}}dx = \int_0^1 (7x^3+1)^{\frac{-1}{3}} \cdot x^2 dx$$

$t = 7x^3 + 1 \qquad dt = 21x^2\,dx \qquad \frac{1}{21}\,dt = x^2\,dx$

$x = 0 \quad \rightarrow \quad t = 1$

$x = 1 \quad \rightarrow \quad t = 7 + 1 = 8$

$$I = \frac{1}{21}\int_1^8 t^{\frac{-1}{3}}\,dt = \frac{1}{21}\left[\frac{3}{2}\,t^{\frac{2}{3}}\right]_1^8 = \frac{1}{21}\cdot\frac{3}{2}\left[(8)^{\frac{2}{3}} - 1\right] = \frac{1}{14}\,(4-1) = \frac{3}{14}$$

Example 6	Evaluate $\int_0^1 (x^3+2)^{10}\,x^5\,dx$

Solution

$$I = \int_0^1 (x^3+2)^{10}\,x^5\,dx = \int_0^1 (x^3+2)^{10}\,x^3 \cdot x^2 dx$$

$t = x^3 + 2 \qquad dt = 3x^2\ dx \qquad \frac{1}{3}\,dt = x^2\ dx$

$x^3 = t - 2$

$x = 0 \quad \rightarrow \quad t = 2$

$x = 1 \quad \rightarrow \quad t = 3$

$$I = \frac{1}{3}\int_2^3 t^{10}(t-2)\,dt = \frac{1}{3}\int_2^3 [t^{11} - 2t^{10}]\,dt = \frac{1}{3}\left[\frac{1}{12}\,t^{12} - \frac{2}{11}\,t^{11}\right]_2^3$$

$$= \frac{1}{3}\left[\left(\frac{3^{12}}{12} - \frac{2\cdot 3^{11}}{11}\right) - \left(\frac{2^{12}}{12} - \frac{2^{12}}{11}\right)\right]$$

Example 7	Evaluate the following integral $\int_{2}^{8} \lvert 2x-10\rvert dx$

Solution

$$I=\int_{2}^{8}\lvert 2x-10\rvert dx=-\int_{2}^{5}(2x-10)\,dx+\int_{5}^{8}(2x-10)\,dx$$

		5	
$2x-10$	$-$		$+$

(zero at 5)

$$=-\Big[x^2-10x\Big]_{2}^{5}+\Big[x^2-10x\Big]_{5}^{8}$$

$$=-[25-50-(4-20)]+[64-80-(25-50)]=-(-25+16)+(-16+25)$$

$$=25-16-16+25=9+9=18$$

Example 8 13 February 19, 1995	Evaluate $\int_{0}^{3}\lvert x^2-x-2\rvert dx$

Solution

$$x^2-x-2=(x-2)(x+1)$$

		-1		2		
$x-2$	$-$		$-$		$+$	2
$x+1$	$-$		$+$		$+$	-1
	$\oplus$		$\ominus$		$\oplus$	

$$I=\int_{0}^{3}\lvert x^2-x-2\rvert dx$$

$$=\int_{0}^{2}-(x^2-x-2)\,dx+\int_{2}^{3}(x^2-x-2)\,dx$$

$$=-\left[\frac{1}{3}x^3-\frac{1}{2}x^2-2x\right]_{0}^{2}+\left[\frac{1}{3}x^3-\frac{1}{2}x^2-2x\right]_{2}^{3}$$

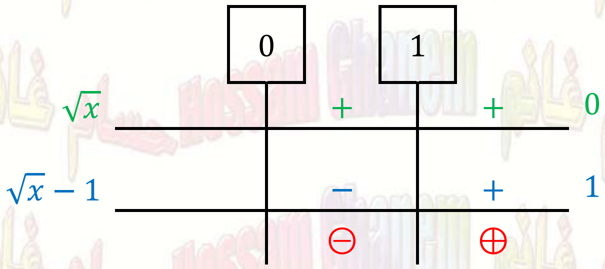
$$=-\left(\frac{8}{3}-2-4\right)+\left[\left(9-\frac{9}{2}-6\right)-\left(\frac{8}{3}-2-4\right)\right]=-\left(\frac{8}{3}-6\right)+\left(9-\frac{9}{2}-6\right)-\left(\frac{8}{3}-6\right)$$

$$=-\frac{8}{3}+6+9-\frac{9}{2}-6-\frac{8}{3}+6=-\frac{16}{3}+9-\frac{9}{2}+6=15-\frac{16}{3}-\frac{9}{2}=\frac{90-32-27}{6}=\frac{31}{6}$$

Example 9 15 February 12, 1996	Evaluate the following integral $\int_0^2 \left\| x-\sqrt{x} \right\| dx$

Solution

$x-\sqrt{x} = \sqrt{x}(\sqrt{x}-1)$



$$I = \int_0^2 |x-\sqrt{x}|dx$$

$$= \int_0^1 -(x-\sqrt{x})dx + \int_1^2 (x-\sqrt{x})dx$$

$$= -\int_0^1 \left(x - x^{\frac{1}{2}}\right)dx + \int_1^2 \left(x - x^{\frac{1}{2}}\right)dx$$

$$= -\left[\frac{1}{2}x^2 - \frac{2}{3}x^{\frac{3}{2}}\right]_0^1 + \left[\frac{1}{2}x^2 - \frac{2}{3}x^{\frac{3}{2}}\right]_1^2$$

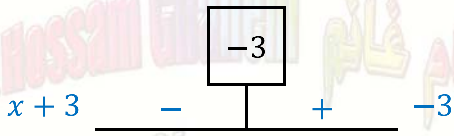
$$= -\left[\frac{1}{2} - \frac{2}{3} - 0\right] + \left(2 - \frac{2}{3}\cdot 2^{\frac{3}{2}}\right) - \left(\frac{1}{2} - \frac{2}{3}\right) = -\frac{1}{2} + \frac{2}{3} + 2 - \frac{2}{3}(2\sqrt{2}) - \frac{1}{2} + \frac{2}{3} = 1 + \frac{4}{3} - \frac{4}{3}\sqrt{2}$$

$$= \frac{7}{3} - \frac{4}{3}\sqrt{2} = \frac{1}{3}(7-4\sqrt{2})$$

Example 10	Evaluate $\int_{-27}^{1} x^{-\frac{2}{3}}\sqrt{x^2+6x+9}\ dx$

Solution

$\sqrt{x^2+6x+9} = \sqrt{(x+3)^2} = |x+3|$



$$I = \int_{-27}^{1} x^{-\frac{2}{3}}\sqrt{x^2+6x+9}\ dx = \int_{-27}^{1} x^{-\frac{2}{3}}|x+3|\ dx$$

$$= -\int_{-27}^{-3} x^{-\frac{2}{3}}(x+3)\ dx + \int_{-3}^{1} x^{-\frac{2}{3}}(x+3)\ dx$$

$$= -\int_{-27}^{-3}\left(x^{\frac{1}{3}} + 3x^{\frac{-2}{3}}\right)dx + \int_{-3}^{1}\left(x^{\frac{1}{3}} + 3x^{\frac{-2}{3}}\right)dx$$

$$= -\left[\frac{3}{4}x^{\frac{4}{3}} + 3\cdot 3\,x^{\frac{1}{3}}\right]_{-27}^{-3} + \left[\frac{3}{4}x^{\frac{4}{3}} + 9x^{\frac{1}{3}}\right]_{-3}^{1}$$

$$= -\left[\frac{3}{4}(-3)^{\frac{4}{3}} + 9(-3)^{\frac{1}{3}}\right] + \left[\frac{3}{4}(-3)^4 - 9(-3)\right] + \left[\frac{3}{4} + 9\right] - \left[\frac{3}{4}(-3)^{\frac{4}{3}} - 9(-3)^{\frac{1}{3}}\right]$$

Example 11 32 August 02, 2008	Find $\int_{0}^{2} \frac{x^2}{\sqrt{2x^3+9}}\,dx$

Solution

$$I = \int_{0}^{2} \frac{x^2}{\sqrt{2x^3+9}}\,dx = \int_{0}^{2} (2x^3+9)^{\frac{-1}{2}}\; x^2\,dx$$

$t = 2x^3 + 9 \quad ⇨⇨ \; dt = \; 6x^2\,dx \qquad ⇨⇨ \; \frac{1}{6}dt = x^2\,dx$

$x = 0 \;\; \rightarrow \;\; t = 9$

$x = 2 \;\; \rightarrow \;\; t = 16 + 9 = 25$

$$I = \frac{1}{6}\int_{9}^{25} t^{\frac{-1}{2}}\,dt = \frac{1}{6}\left[2t^{\frac{1}{2}}\right]_{9}^{25} = \frac{2}{6}[5-3] = \frac{2}{6}\cdot 2 = \frac{2}{3}$$

Homework

1	Evaluate	$\int_{1}^{4} \frac{x^2 - 1}{\sqrt{x}} dx$	
2	Evaluate	$\int_{1}^{2} x^2 \sqrt{x^3 - 5}\, dx$	
3	Evaluate	$\int_{0}^{2} x(x^2 - 2)^5\, dx$	29 June 4. 2007
4	Find	$\int_{0}^{1} (x^2 - 1)^{10}\, x^3\, dx$	12 January 9, 1995
5	Evaluate	$\int_{-1}^{1} \frac{x^2\ dx}{\sqrt{5x^3 + 8}}$	
6	Evaluate	$\int_{0}^{2} \frac{5x^2}{\sqrt{3x^2 + 9}} dx$	27 May 30. 2006
7	Evaluate	$\int_{-1}^{1} \frac{x}{\sqrt{5 - x^2}} dx$	19 July 29, 2000
8	Evaluate	$\int_{0}^{6} \lvert x - 3 \rvert dx$	14 January 6, 1996

Homework

9	Evaluate	$\int_{1}^{3} \lvert x-2 \rvert dx$	18 May 24 ,2000
10	Evaluate	$\int_{0}^{2} \lvert x^2+3x-4 \rvert\ dx$	11 August 11, 1994 A
11	Evaluate	$\int_{0}^{3} \lvert x^2+2x-3 \rvert dx$	7 June 17, 1993
12	Evaluate	$\int_{-1}^{1} \lvert x^2+x-2 \rvert dx$	
13	Evaluate	$\int_{0}^{2} \lvert x^3-x^2 \rvert dx$	6 January 6, 1993
14	Find	$\int_{-8}^{8} x^{-\frac{2}{3}} \sqrt{x^2-2x+1}\ dx$	
15	Find	$\int_{0}^{1} \frac{2x^2-x-1}{2x+1} dx$	34 June 21, 2009
16	Evaluate	$\int_{1}^{8} \frac{x^2+3}{\sqrt[3]{x}} dx$	